Computing Humbert Surfaces

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The Siegel upper half plane

Definition
The Siegel upper half plane of degree $g$ is

$$\mathcal{H}_g = \{ \tau \in \text{Mat}_{g \times g}(\mathbb{C}) \mid t\tau = \tau, \quad \text{Im}(\tau) > 0 \}.$$  

- Each $\tau \in \mathcal{H}_g$ corresponds to a PPAV $A_{\tau}/\mathbb{C}$ with period matrix $(\tau \ I_g) \in \text{Mat}_{g \times 2g}(\mathbb{C})$.
- $A_{\tau} \cong A_{\tau'} \iff \exists M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{Z})$ such that $\tau' = M \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$.
- $A_g = \text{Sp}_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g$ is a moduli space for dimension $g$ PPAV’s.
- $\dim A_g = \frac{1}{2}g(g + 1)$. In particular, $\dim A_2 = 3$ and $A_2$ is called the Siegel modular threefold.
Extra endomorphisms

Let $A$ be a PPAS ($g = 2$). Then $\text{End}(A)$ is an order in $\text{End}(A) \otimes \mathbb{Q}$ which isomorphic to one of the following algebras:

1. quartic CM field
2. indefinite quaternion algebra over $\mathbb{Q}$
3. real quadratic field
4. $\mathbb{Q}$

The irreducible components of the corresponding moduli spaces in $\mathcal{A}_2$ which have “extra endomorphisms” are known as

1. CM points
2. Shimura curves
3. Humbert surfaces
Humbert’s equation

Humbert showed that any \( \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in A_2 \) satisfying the equation

\[
k\tau_1 + \ell\tau_2 - \tau_3 = 0
\]

defines a Humbert surface \( H_\Delta \) of discriminant \( \Delta = 4k + \ell > 0 \).

**Example**

\( H_1 = \text{Sp}_4(\mathbb{Z}) \setminus \left\{ \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_3 \end{pmatrix} \right\} = \text{Sp}_4(\mathbb{Z}) \setminus \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \right\} \), the set of abelian varieties which split as a product of elliptic curves.

**Task:** Find “useful” algebraic models for \( H_\Delta \).
Torelli says that the map $C \mapsto \text{Jac}(C)$ defines a birational map between $\mathcal{M}_2$, the moduli space of genus 2 curves and $\mathcal{A}_2$ (In fact $\mathcal{M}_2 \cong \mathcal{A}_2 - H_1$).

The function field of $\mathcal{A}_2$ (and hence $\mathcal{M}_2$) is $\mathbb{C}(j_1, j_2, j_3)$ where $j_i$ are the absolute Igusa invariants.

There exists an irreducible polynomial $H_\Delta(j_1, j_2, j_3)$ whose zero set is the Humbert surface of discriminant $\Delta$.

Unfortunately, working with $j_i$ is impractical (enormous degrees, giant coefficients).

Solution: add some level structure.
Runge uses level $\Gamma^*(2, 4)$ structure, with four theta functions:

$$\theta \left[ \begin{array}{c} a \\ (0, 0) \end{array} \right] (2\tau), \ a \in \mathbb{Z}^2/2\mathbb{Z}^2$$

where

$$\theta \left[ \begin{array}{c} m' \\ m'' \end{array} \right] (\tau) = \sum_{x \in \mathbb{Z}^2} e^{2\pi i \left( \frac{1}{2} (x + \frac{m'}{2}) \cdot \tau \cdot t (x + \frac{m'}{2}) + (x + \frac{m'}{2}) \cdot t (\frac{m''}{2}) \right)}$$

are classical theta functions of half integral characteristics determined by values $m', m'' \in \mathbb{Z}^2/2\mathbb{Z}^2$. 
Algebraic models

Rosenhain model

We use level $\Gamma(2)$-structure with three functions

\[
\lambda_1(\tau) = \left( \frac{\theta[0 0] \theta[1 0]}{\theta[0 1] \theta[0 1]} \right)^2, \\
\lambda_2(\tau) = \left( \frac{\theta[0 0] \theta[1 1]}{\theta[0 1] \theta[1 1]} \right)^2, \\
\lambda_3(\tau) = \left( \frac{\theta[0 0] \theta[1 1]}{\theta[1 1] \theta[1 1]} \right)^2
\]

called Rosenhain invariants. These generate the function field of $A_2(2) = \Gamma(2) \backslash \mathcal{H}_2$. 
Relation to genus 2 curves

- Given a genus 2 curve $C : y^2 = \prod_{i=1}^{6}(x - u_i)$ we can send three of the $u_i$ to $0, 1, \infty$ via a fractional linear transformation to get an isomorphic curve with a Rosenhain model

$$y^2 = x(x - 1)(x - t_1)(x - t_2)(x - t_3).$$

The $t_i$ are called Rosenhain invariants.

- $(0, 1, \infty, t_1, t_2, t_3)$ determines an ordering of the Weierstrass points and a level 2 structure on $\text{Jac}(C) (\in A_2(2))$.

- Let $M_2(2)$ denote the moduli space of genus 2 curves together with a full level 2 structure. Points of $M_2(2)$ are given by triples $(t_1, t_2, t_3)$ where $t_i \neq t_j, 0, 1$ for all $i, j$.

- The forgetful morphism $M_2(2) \rightarrow M_2$ is a Galois covering of degree $720 = |S_6|$ where $S_6$ acts on the Weierstrass 6-tuple by permutations, followed by renormalising the first three to $(0, 1, \infty)$. 

Runge’s method

Let $\phi : \mathcal{A}' \to \mathcal{A}_2$ be a finite cover of $\mathcal{A}_2$. Then

$$\phi^{-1}H_\Delta = \bigcup_{\text{finite}} H_\Delta^{(i)}.$$

Given functions $\{f_i(\tau)\}_{i=1,...,n}$ generating the function field of $\mathcal{A}'$, compute $H_\Delta^{(i)}(f_1, \ldots, f_n)$ as follows:

1. Calculate the degree of the Humbert components $H_\Delta^{(i)}$ (given by a formula).

2. Compute power series representations of the $f_i(\tau)$ restricted to $H_\Delta \subset \mathcal{H}_2$.

3. Solve $H_\Delta^{(i)}(f_1, \ldots, f_n) = 0$ in the power series ring (truncated series with large precision) using linear algebra.

We shall consider the level 2 covering $\mathcal{A}_2(2) \to \mathcal{A}_2$. 
Fortunately much arithmetic-geometric information is known about Humbert surfaces (van der Geer '82). The number of Humbert components in $A_2(2)$ is

$$m(\Delta) = \begin{cases} 
10 & \text{if } \Delta \equiv 1 \mod 8 \\
15 & \text{if } \Delta \equiv 0 \mod 4 \\
6 & \text{if } \Delta \equiv 5 \mod 8
\end{cases}$$

(see Besser '98).
The degree of any Humbert component $H^{(i)}_{\Delta}$ in $\mathcal{A}_2(2)$ is given by a recursive formula

$$a_{\Delta} = \sum_{x>0} v(\Delta/x^2)m(\Delta/x^2)\deg \left( H^{(i)}_{\Delta/x^2} \right)$$

where

$$v(x) = \begin{cases} 
1/2 & \text{if } x = 1 \\
1 & \text{if } x \geq 2, x \equiv 0, 1 \mod 4 \\
0 & \text{otherwise}
\end{cases}$$

Moreover, $a_{\Delta}$ is the coefficient of a certain modular form of weight $5/2$ for the group $\Gamma_0(4)$, which fortunately has a more elementary description due to a formula of Siegel:

$$a_{\Delta} - 24 \sum_{x \in \mathbb{Z}} \sigma_1 \left( \frac{\Delta - x^2}{4} \right) = \begin{cases} 
12\Delta - 2 & \text{if } \Delta = \square \\
0 & \text{otherwise}
\end{cases}$$
Here are the degrees for small discriminants:

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>1</th>
<th>4</th>
<th>5</th>
<th>8</th>
<th>9</th>
<th>12</th>
<th>13</th>
<th>16</th>
<th>17</th>
<th>20</th>
<th>21</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{deg}(H_{\Delta}^{(i)})$</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>24</td>
<td>16</td>
<td>40</td>
<td>32</td>
<td>48</td>
<td>32</td>
<td>80</td>
<td>48</td>
</tr>
<tr>
<td>actual deg</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>8</td>
<td>16</td>
<td>16</td>
<td>40</td>
<td>24</td>
<td>48</td>
<td>32</td>
<td>80</td>
<td>48</td>
</tr>
</tbody>
</table>

Remarks

- When $\Delta \equiv 0 \pmod{4}$ we have
  
  $$m_{\text{Runge}}(\Delta) = 4 \times m_{\text{Rosenhain}}(\Delta)$$

  $$\Rightarrow \text{deg}_{\text{Runge}}(\Delta) = \frac{1}{4} \times \text{deg}_{\text{Rosenhain}}(\Delta).$$

- In reality (after computing these equations) the actual degrees of $H_{\Delta}^{(i)}(\lambda_j)$ are less than what the formula produces. For example
  
  $$H_1 : \{e_i - e_j = 0, i \neq j\} \cup \{e_i = 0\} \cup \{e_i - 1 = 0\}.$$
Write $\Delta = 4k + \ell$ where $\ell$ is either 0 or 1, and $k$ is uniquely determined. The Humbert surface of discriminant $\Delta$ can be defined by the set

$$H_\Delta = \text{Sp}_4(\mathbb{Z}) \setminus \left\{ \left( \begin{array}{cc} \tau_1 & \tau_2 \\ \tau_2 & k\tau_1 + \ell\tau_2 \end{array} \right) \in \mathcal{H}_2 \right\}.$$ 

Restrict $\theta \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$ to $H_\Delta$ to get a Laurent series

$$\theta \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right](\tau) = \sum_{(x_1, x_2) \in \mathbb{Z}^2} e^{\pi i (x_1 c + x_2 d)} r (2x_1 + a)^2 + k(2x_2 + b)^2 q^2 (2x_1 + a)(2x_2 + b) + \ell(2x_2 + b)^2$$

where $r = e^{2\pi i \tau_1/8}$ and $q = e^{2\pi i \tau_2/8}$. 
Unfortunately $q$ has negative exponents. Substitute $r = pq$ to get

\[ \sum_{(x_1, x_2) \in \mathbb{Z}^2} (-1)^{x_1 c + x_2 d} p^{(2x_1+a)^2+k(2x_2+b)^2} q^{(2x_1+a+2x_2+b)^2+(k+\ell-1)(2x_2+b)^2} \]

which is a power series with integer coefficients.

Using this representation one can compute the restriction of $\lambda_1, \lambda_2, \lambda_3$ to a Humbert surface as elements of $\mathbb{Z}[[p, q]]/(p^N, q^N)$ fairly easily.
Let $d = \deg(H_{\Delta}^{(i)})$. To find the algebraic relation $H_{\Delta}^{(i)}$:

▶ Compute all monomials of degree $\leq d$ in the variables $e_1, e_2, e_3$.

▶ Substitute $e_i = \lambda_i(p, q) \in \mathbb{Z}[[p, q]]/(p^N, q^N)$ in each monomial.

▶ Use linear algebra to find linear dependencies between the power series monomials $p^m q^n$ (compute null space of a big matrix).
With high enough precision there will be exactly one linear relation between the monomials $e_i$. This produces the polynomial relation $H^{(i)}_{\Delta}(e_1, e_2, e_3) = 0$ which defines a Humbert component.

Once one component has been determined, the others can easily be found by looking at the Rosenhain ($S_6$) orbit of a component.

These other components will turn out to be useful when we look at Shimura curves.
Runtime analysis

- There are:
  - \(\binom{d+3}{3} = O(d^3)\) monomials to be evaluated
  - \(O(N^2)\) coefficients of evaluated power series expressions of precision \(N\).

- Runtime cost is dominated by the nullspace calculation: \(O(d^6 N^2) \geq O(d^9)\) to find a unique solution.

- Symmetries of the equation (arising from the fixed group of the humbert component) can be exploited to reduce the matrix size by a constant factor, giving a speedup by a constant factor.

- Not overly efficient, but at least it’s only a one time calculation.
Example

We calculate a component of $H_5$:

\[
\begin{align*}
\lambda_1 &= 1 + 16p^4q^8 + O(p^{12}q^{12}) \\
\lambda_2 &= 1 + 4q^4 + 8q^8 - 8p^4q^4 - 24p^4q^8 + 4p^8q^8 + 48p^8q^8 + O(p^{12}q^{12}) \\
\lambda_3 &= 1 + 4q^4 + 8q^8 + 8p^4q^4 + 40p^4q^8 + 4p^8q^8 + 48p^8q^8 + O(p^{12}q^{12})
\end{align*}
\]

Using power series with precision 65, we compute the Humbert component

\[
\begin{align*}
e_2^2e_3^2 - 2e_2^2e_3^3 + e_2^2e_3^4 + 2e_1e_2e_3^3 - 2e_1e_2e_3^4 - 2e_1e_2e_3^2 - 2e_1e_2e_3^2 + 4e_1e_2^2e_3^3 - 2e_1e_2e_3^2 \\
- 4e_1e_2^2e_3^3 - 2e_1e_2e_3^2 - 2e_1e_2e_3^2 + 4e_1^2e_2e_3^3 + 2e_1e_2e_3^2 + 2e_1e_2e_3^2  \\
+ 4e_1^2e_2e_3^2 + e_1^2e_2^2 - 2e_1^2e_2e_3^3 + e_1^2e_2^2e_3^2 - 2e_1^2e_3^3 - 2e_1^2e_2e_3 + 4e_1^2e_2e_3^2 - 2e_1^2e_2e_3^2 - 2e_1^2e_2e_3^2 \\
- 2e_1^3e_2^2e_3 - 2e_1^3e_2e_3^2 + e_1^3e_3^2 - 2e_1^3e_2e_3^2 + e_1^3e_2e_3^2 - 2e_1^3e_2e_3^2 + e_1^3e_2e_3^2
\end{align*}
\]
Application: Computing Shimura Curves
Let $R$ be an order in an indefinite $\mathbb{Q}$-quaternion algebra $A$.

- $R$ is a QM-order if $R = \text{End}(X)$ for some abelian surface $X$.

- Any $x \in A$ satisfies $x^2 - tx + n = 0$ where $t, n$ are the reduced trace, norm respectively.

- $\Delta(x) = t(x)^2 - 4n(x)$ defines a discriminant form
  \[
  \Delta(x, y) = \frac{1}{2}(\Delta(x + y) - \Delta(x) - \Delta(y))
  .
  \]

- The discriminant $d(x_1, \ldots, x_4)$ of a module generated by $x_1, \ldots, x_4$ is defined to be the positive square root of
  \[
  d(x_1, \ldots, x_4)^2 = -\det(t(x_i x_j))
  .
  \]
Theorem (Runge ’99)

1. Any QM-order can be written as $R = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\alpha\beta$ such that the discriminant matrix

$$S_\Delta = \begin{pmatrix} \Delta(\alpha) & \Delta(\alpha, \beta) \\ \Delta(\alpha, \beta) & \Delta(\beta) \end{pmatrix}$$

is positive definite. The discriminant of $R$ equals $\det(S_\Delta)/4$.

2. A change of basis corresponds to changing the discriminant matrix to $t_g S_\Delta g$ for some $g \in \text{GL}_2(\mathbb{Z})$. ⇒ can assume discriminant matrix is reduced.

3. If two orders have the same discriminant matrix which is primitive (gcd(entries) = 1) then the corresponding Shimura curves are isomorphic.
Theorem (Hashimoto '95, Runge '99)

Let $\mathcal{O} = \mathbb{Z}[\omega]$ be a quadratic order of discriminant $\Delta$. Let $S_\Delta$ be a discriminant matrix of a QM order $R$. The following are equivalent:

1. $\Delta$ is primitively represented by $S_\Delta$.
2. There exists an embedding $\mathcal{O} \hookrightarrow R$ such that $R \cap \mathbb{Q}(\omega) = \mathcal{O}$.
3. A Shimura curve $\mathcal{C}$ with QM order $R$ is contained in $H_\Delta$.

If we work in a finite cover, we have

$$\mathcal{C}^{(h)} \subset H^{(i)}_{\Delta(\alpha)} \cap H^{(j)}_{\Delta(\beta)}$$

if and only if we can write

$$^{t}g S_\Delta g = \begin{pmatrix} \Delta(\alpha) & * \\ * & \Delta(\beta) \end{pmatrix}$$

for some $g \in \text{GL}_2(\mathbb{Z})$. 
Example

\( H_5 \cap H_8 \) contains four Shimura curves \( C_S \):

<table>
<thead>
<tr>
<th>Discriminant matrix ( S )</th>
<th>QM-order discriminant ( = \det(S)/4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\begin{pmatrix} 5 &amp; 0 \ 0 &amp; 8 \end{pmatrix}))</td>
<td>10</td>
</tr>
<tr>
<td>((\begin{pmatrix} 5 &amp; 2 \ 2 &amp; 8 \end{pmatrix}))</td>
<td>9</td>
</tr>
<tr>
<td>((\begin{pmatrix} 5 &amp; 4 \ 4 &amp; 8 \end{pmatrix}) \sim (\begin{pmatrix} 5 &amp; 1 \ 1 &amp; 5 \end{pmatrix}))</td>
<td>6</td>
</tr>
<tr>
<td>((\begin{pmatrix} 5 &amp; 6 \ 6 &amp; 8 \end{pmatrix}) \sim (\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 4 \end{pmatrix}))</td>
<td>4</td>
</tr>
</tbody>
</table>

- This intersection was first computed by Hashimoto and Murabayashi (1995).
Computing intersections

- For intersections of Humbert components $H_{\Delta_i}(e_1, e_2, e_3)$ we can find plane affine models simply by taking resultants with respect to $e_1$.
- As we are working with coordinates in $\mathcal{M}_2(2) = \mathcal{A}_2(2) - H_1$, we will not be able to compute any Shimura curves in $H_1$.
- $S_6$ acts on Humbert components, hence acts on their intersections producing isomorphic curves.
- Take one curve from each $S_6$-orbit. Each of these intersections is a component of a Shimura curve $C_S$ for some discriminant matrix $S$. 
Example

In our $H_5 \cap H_8$ example, there are three non-equivalent intersections:

\[ C_1 : \text{a genus 1 curve} \]
\[ C_2 : \text{a genus 3 hyperelliptic curve} \]
\[ C_3 : \text{a genus 3 non-hyperelliptic curve} \]

and the Shimura curves in $\mathcal{M}_2(2)$ are

\[ C_{\binom{5}{0}0}, C_{\binom{5}{2}2} \text{ and } C_{\binom{5}{1}1} \]

so there is a one-one correspondence between the $C_i$ and the $C_S$, to be determined.
Look at other Humbert intersections with “related” discriminants. Write $\mathcal{D}(a, b)$ for the set of discriminant matrices of QM-orders of Shimura curves in $H_a \cap H_b$. We have

\[
\begin{align*}
\mathcal{D}(5, 5) &= \{ (\begin{array}{cc} 5 & 1 \\ 1 & 5 \end{array}), (\begin{array}{cc} 4 & 2 \\ 2 & 5 \end{array}) \} \\
\mathcal{D}(4, 5) &= \{ (\begin{array}{cc} 4 & 0 \\ 0 & 5 \end{array}), (\begin{array}{cc} 4 & 2 \\ 2 & 5 \end{array}) \} \cup \{ (\begin{array}{cc} 1 & 0 \\ 0 & 4 \end{array}) \} \\
\mathcal{D}(5, 9) &= \{ (\begin{array}{cc} 5 & 1 \\ 1 & 9 \end{array}), (\begin{array}{cc} 5 & 2 \\ 2 & 8 \end{array}), (\begin{array}{cc} 4 & 0 \\ 0 & 5 \end{array}) \} \cup \{ (\begin{array}{cc} 1 & 0 \\ 0 & 4 \end{array}) \} \\
\mathcal{D}(5, 8) &= \{ (\begin{array}{cc} 5 & 0 \\ 0 & 8 \end{array}), (\begin{array}{cc} 5 & 2 \\ 2 & 8 \end{array}), (\begin{array}{cc} 5 & 1 \\ 1 & 5 \end{array}) \} \cup \{ (\begin{array}{cc} 1 & 0 \\ 0 & 4 \end{array}) \}
\end{align*}
\]

- Since $\mathcal{D}(4, 5) \cap \mathcal{D}(5, 5) = \{ (\begin{array}{cc} 4 & 2 \\ 2 & 5 \end{array}) \}$ we can identify the corresponding curve. Hence we also know $(\begin{array}{cc} 5 & 1 \\ 1 & 5 \end{array})$ (and $(\begin{array}{cc} 4 & 0 \\ 0 & 5 \end{array})$).
- Similarly $(\begin{array}{cc} 5 & 2 \\ 2 & 8 \end{array})$ can be matched by $\mathcal{D}(5, 8) \cap \mathcal{D}(5, 9) = \{ (\begin{array}{cc} 5 & 2 \\ 2 & 8 \end{array}) \}$. 
In the end we find that:

\[ C_{\begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}} : \text{ the genus 1 curve} \]
\[ C_{\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}} : \text{ the genus 3 hyperelliptic curve} \]
\[ C_{\begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix}} : \text{ the genus 3 non-hyperelliptic curve} \]

Thanks for listening!